Proposed Solution to ΠME Journal Problem Department #1388

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1 Problem #1388

We place the numbers that are 1 modulo 8 in a matrix as follows: we start with 1 in the upper left corner, then in the next diagonal put 9 then 17, in the next diagonal 25, 33, and 41, and so on. We show its start:

$$\begin{bmatrix} 1 & 9 & 25 & 49 & 81 & \ldots \\ 17 & 33 & 57 & 89 & 129 & \ldots \\ 41 & 65 & 97 & 137 & 185 & \ldots \\ 73 & 105 & 145 & 193 & 249 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Prove or disprove that the sum of any two consecutive diagonals, running upper right to bottom left is a perfect cube. For example,

$$(1) + (9 + 17) = 27 = 3^3, \quad (9 + 17) + (25 + 33 + 41) = 125 = 5^3$$

If you cannot prove or disprove the conjecture, for how many diagonals can you confirm it?

2 Suggested Solution to Problem #1388

PROOF: First notice that by subtracting 1 from the top row of the matrix, we get a sequence of numbers that all have 8 as a common factor:

$$0 = 0 * 8, \quad 8 = 1 * 8, \quad 24 = 3 * 8, \quad 48 = 6 * 8, \quad 80 = 10 * 8, \ldots$$

In particular, the other factor represents the number of entries in each diagonal prior. Taking into account the factor of 8 and the 1 we subtracted at the beginning, we get the following formula for each number in the top row of the original matrix where n is the column:

$$1 + 8 * \sum_{i=1}^{n} i - 1 = 1 + 8 * \frac{n^2 - n}{2} = 4n^2 - 4n + 1 = (2n-1)^2$$

Thus the top row of the matrix is the sequence of odd numbered perfect squares:

$$1^2 = 1, \quad 3^2 = 9, \quad 5^2 = 25, \quad 7^2 = 49, \quad 9^2 = 81, \ldots$$

Also since each diagonal (starting with the number at the top of the n^{th} column) has exactly n entries (with indexing starting at 1), we can sum any individual diagonal n, by starting with the square of the corresponding n^{th} odd number $(2n-1)^2$.

From there we add 8 to get each consecutive number in the diagonal, continuing until we have n numbers for the diagonal. With these considerations we get the following sum for diagonal n:

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} (2n-1)^2 + 8 * (i-1) = \sum_{i=1}^{n} 4n^2 - 4n + 8i - 7 \\ &= 4n^3 - 4n^2 - 7n + 4n^2 - 4n = 4n^3 - 3n \end{aligned}$$

If we then add the sums for diagonals n and $n+1$ we get:

$$\begin{aligned} S(n) + S(n+1) &= (4n^3 - 3n) + (4(n+1)^3 - 3(n+1)) \\ &= (4n^3 - 3n) + (4n^3 + 12n^2 + 12n + 4 - 3n - 3) = (4n^3 - 3n) + (4n^3 + 12n^2 + 9n + 1) \\ &= 8n^3 + 12n^2 + 6n + 1 = (2n+1)^3 \end{aligned}$$

Therefore the sum of any 2 consecutive diagonals, n and $n+1$, is a perfect cube, in particular the cube of the n^{th} positive odd number.

QED